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Positive Spline Operators and Orthogonal Splines

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We study some convergence (in L^p) and spectral properties of the positive spline operators $U_{n,k}f(x)=\sum_i (\int_0^1 M_{i,k}(t)f(t)\,dt)\,N_{i,k}(x)$, where $\sum_i N_{i,k}=1$ and $\int_0^1 M_{i,k}=1$, $N_{i,k}$ and $M_{i,k}$ being the B-splines of degree k and class C^{k-1} associated with some partition of $I=[0,1]$ into n subintervals. Their eigenfunctions are orthogonal splines generalizing in some sense the Legendre polynomials with which they share many properties. © 1988 Academic Press, Inc.

I. INTRODUCTION

Let $\Delta_n=\{0=x_0<x_1<\cdots<x_n=1\}$ be an arbitrary partition of $I=[0,1]$ extended to a sequence $\Delta_{n,k}$ of *knots* by setting $x_{-k}=\cdots=x_{-1}=x_0$ and $x_n=\cdots=x_{n+k}=1$ $(n,\ k\geqslant 1)$. The *nodes* of $\Delta_{n,k}$ are the points

$$\xi_{i,k}=(x_{i+1}+\cdots+x_{i+k})/k \qquad (-k\leqslant i\leqslant n-1).$$

The *normalized B-splines of degree k* are defined by

$$N_{i,k}(x)=\frac{x_{i+k+1}-x_i}{k+1}M_{i,k}(x),$$

where the B-spline $M_{i,k}(x)$ is the $(k+1)$th divided difference of the function $(k+1)(\cdot-x)_+^k$ with respect to $x_i,\ \ldots,\ x_{i+k+1}$.

It is well known that these B-splines verify

$$N_{i,k}(x)\geqslant 0, \qquad \operatorname{supp}(N_{i,k})=[x_i,\ x_{i+k+1}],$$

$$\sum_{i=-k}^{n-1} N_{i,k}(x)=1, \qquad \sum_{i=-k}^{n-1}\xi_{i,k}N_{i,k}(x)=x, \qquad \text{and} \qquad \int_0^1 M_{i,k}(x)\,dx=1.$$

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Schoenberg [26] constructed a generalization of classical Bernstein operators by setting, for $f \in C(I)$,

$$S_{n,k}f(x) = \sum_{i=-k}^{n-1} f(\xi_{i,k})\, N_{i,k}(x). \tag{1}$$

Their properties have been studied in [17].

More recently, Müller [18] defined a second family of spline operators, generalizing the Bernstein–Kantorovitch operators [14], by setting, for $f \in L^p(I)$,

$$T_{n,k}f(x) = \sum_{i=-k}^{n-1} \left(\int_{\xi_{i-1,k+1}}^{\xi_{i,k+1}} f(t)\, dt \right) M_{i,k}(x). \tag{2}$$

Here, we want to study a third family of positive spline operators, generalizing the *modified Bernstein operators* of Durrmeyer [11] and Derriennic [8, 9] which are defined, for $f \in L^p(I)$, by

$$\tilde{B}_k f(x) = (k+1) \sum_{i=0}^{k} \left(\int_0^1 f(t)\, b_{ik}(t)\, dt \right) b_{ik}(x), \tag{3}$$

where $b_{i,k}(x) = \binom{k}{i} x^i (1-x)^{k-i}$ for $0 \leqslant i \leqslant k$.

A remarkable property of these operators is that *their eigenfunctions are the Legendre polynomials* on [0, 1]. This property is true, more generally, for similar operators associated with the Jacobi and Laguerre polynomials [4, 22]. It seemed rather natural to extend this property to the following spline operators

$$U_{n,k}f(x) = \sum_{i=-k}^{n-1} \langle M_{i,k}, f \rangle\, N_{i,k}(x), \tag{4}$$

where $\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$.

We give some convergence results when $|\Delta_n| = \max_i (x_{i+1} - x_i)$ tends to zero as $n \to +\infty$ and $f \in L^p(I)$ $(1 \leqslant p \leqslant +\infty)$.

Then we prove that the eigenfunctions of (4) form a basis of orthogonal splines on [0, 1], generalizing the Legendre polynomials, and different from those given by Schoenberg [26]. More precise results are given about piecewise linear and quadratic orthogonal splines.

We use the notation $\mathrm{Sp}(k, \Delta_n)$ for the space of splines of degree k, class C^{k-1}, on the partition Δ_n: its dimension is $n+k$ and a basis is $\{N_{i,k}(x), -k \leqslant i \leqslant n-1\}$.

II. Convergence Properties in $L^p(I)$

THEOREM 1. (i) *$U_{n,k}$ is a positive, self-adjoint operator of norm unity in $\mathscr{L}(L^p(I))$.*

(ii) *For every $k \geq 1$, $U_{nk} f$ converges to $f \in L^p(I)$ when $n \to +\infty$ and $|\Delta_n| \to 0$.*

(iii) *When f is integrable, $\int_0^1 U_{nk} f(t)\, dt = \int_0^1 f(t)\, dt$.*

Proof. $U_{nk} f(x) = \int_0^1 K_{n,k}(x, t) f(t)\, dt$, where

$$\begin{aligned}
K_{n,k}(x, t) &= \sum_{i=-k}^{n-1} M_{i,k}(t)\, N_{i,k}(x) \\
&= \sum_i \frac{(x_{i+k+1} - x_i)}{k+1} M_{i,k}(x)\, M_{i,k}(t) \\
&= K_{n,k}(t, x) \geq 0.
\end{aligned}$$

The kernel being positive and symmetric, $U_{n,k}$ is a positive and self-adjoint operator. For $f \in L^p(I)$ and $1/p + 1/q = 1$, we get by Hölder's inequality

$$|U_{nk} f(x)| \leq \left[\int_0^1 K_{n,k}(x, t)\, dt\right]^{1/q} \left[\int_0^1 K_{n,k}(x, t)\, |f(t)|^p\, dt\right]^{1/p}.$$

Since

$$\begin{aligned}
\int_0^1 K_{n,k}(x, t)\, dt &= \sum_i \left(\int_0^1 M_{i,k}(t)\, dt\right) N_{i,k}(x) \\
&= \sum_i N_{i,k}(x) = 1 = \int_0^1 K_{n,k}(x, t)\, dx.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^1 |U_{n,k} f(x)|^p\, dx &\leq \int_0^1 \int_0^1 K_{n,k}(x, t)\, |f(t)|^p\, dt\, dx \\
&= \int_0^1 \left(\int_0^1 K_{n,k}(x, t)\, dx\right) |f(t)|^p\, dt \\
&= \int_0^1 |f(t)|^p\, dt.
\end{aligned}$$

This proves $\|U_{nk} f\|_p \leq \|f\|_p$. But when $f = e_0$ ($e_0(x) = 1$ for all x), $U_{nk} e_0 = e_0$, thus in fact we have an equality and the norm of U_{nk} is 1 in the space $\mathscr{L}(L^p(I))$ of linear continuous operators on $L^p(I)$. For the convergence in $L^p(I)$, we use a Korovkin-type theorem [1, 10] which asserts that for positive linear contractions, it is sufficient to prove it for the two test functions $e_0(x)=1$, $e_1(x)=x$. But we have

$$e_1=\sum_i \xi_{i,k} N_{i,k}, \qquad U_{n,k} e_1=\sum_i \langle M_{i,k}, e_1\rangle N_{i,k},$$

and

$$\langle M_{i,k}, e_1\rangle=\int_0^1 x M_{i,k}(x)\,dx=\left(\sum_{j=0}^{k+1} x_{i+j}\right)\Big/(k+2) \text{ (see, e.g., Neuman [16]).}$$

This gives

$$U_{n,k} e_1-e_1=\frac{2}{k+2} \sum_{i=-k}^{n-1}\left(\frac{x_i+x_{i+k+1}}{2}-\xi_{i,k}\right) N_{i,k}.$$

Since $|x_i-\xi_{i,k}| \leqslant (1/k) \sum_{j=1}^{k} |x_i-x_{i+j}| \leqslant \frac{1}{2}(k+1)|\Delta_n|$, we get at once; for all $k \geqslant 1$,

$$|U_{n,k} e_1(x)-e_1(x)| \leqslant 2\left(\frac{k+1}{k+2}\right)|\Delta_n| \leqslant 2|\Delta_n|.$$

In particular $\|U_{n,k} e_1-e_1\|_p \rightarrow 0$ when $|\Delta_n| \rightarrow 0$. Q.E.D.

Finally

$$\int_0^1 U_{n,k} f(t)\,dt=\langle U_{n,k} f, e_0\rangle=\langle f, U_{n,k} e_0\rangle$$
$$=\langle f, e_0\rangle=\int_0^1 f(t)\,dt. \quad ∎$$

Let $L^{p,1}(I)$ be the space of functions $f \in L^p(I)$ with $f' \in L^p(I)$ and the norm $\|f\|_{p,1}=\|f\|_p+\|f'\|_p$.

LEMMA 1. *For each* $x \in I$,

$$\sum_{i=-k}^{n-1} N_{i,k}(x) \cdot \int_{x_i}^{x_{i+k+1}} M_{i,k}(t)|t-x|\,dt \leqslant (k+1)|\Delta_n|$$

holds.

Proof. Let $A_{i,k}(x)=\int_{x_i}^{x_{i+k+1}} M_{i,k}(t)|t-x|\,dt$ and let us fix $x \in [x_j, x_{j+1}]$, $0 \leqslant j \leqslant n-1$. The only nonzero B-splines at x have indices $i=-k+j, \ldots, j$, hence $t \in [x_{-k+j}, x_{k+j+1}]$, $|t-x| \leqslant (k+1)|\Delta_n|$ and, using the properties of B-splines

$$A_{i,k}(x) \leqslant (k+1)\,\Delta_n \qquad \text{and} \qquad \sum_{i=-k}^{n-1} A_{i,k}(x)\,N_{i,k}(x) \leqslant (k+1)\,|\Delta_n|. \quad \blacksquare$$

LEMMA 2. *For* $f \in L^{p,1}(I)$, $1 \leqslant p \leqslant +\infty$,

$$\|U_{n,k} f - f\|_p \leqslant (k+1)\,\|f'\|_p\,|\Delta_n|$$

holds.

Proof. For x fixed in I, we have

$$|U_{n,k} f(x) - f(x)| \leqslant \sum_{i=-k}^{n-1} N_{i,k}(x) \int_{x_i}^{x_{i+k+1}} M_{i,k}(t) \left(\int_x^t |f'(u)|\,du \right) dt$$

Let $A_{i,k}(x) = \int_{x_i}^{x_{i+k+1}} M_{i,k}(t)\,|t-x|\,dt$ and

$$B_{i,k}(x) = \int_{x_i}^{x_{i+k+1}} M_{i,k}(t) \left(\int_x^t |f'(u)|^p\,du \right) dt.$$

Hölder's inequality $(1/p + 1/q = 1)$ gives successively

$$\int_x^t |f'(u)|\,du \leqslant |t-x|^{1/q} \left(\int_x^t |f'(u)|^p\,du \right)^{1/p},$$

$$\int_{x_i}^{x_{i+k+1}} M_{i,k}(t) \left(\int_x^t |f'(u)|\,du \right) dt \leqslant A_{i,k}^{1/q}(x)\,B_{i,k}^{1/p}(x),$$

$$|U_{n,k} f(x) - f(x)| \leqslant \left[\sum_i A_{i,k}(x)\,N_{i,k}(x) \right]^{1/q} \left[\sum_i B_{i,k}(x)\,N_{i,k}(x) \right]^{1/p}. \tag{5}$$

Lemma 1 gives

$$\sum_i A_{i,k}(x)\,N_{i,k}(x) \leqslant (k+1)\,|\Delta_n|. \tag{6}$$

For $x \in [x_i, x_{i+k+1}]$ and by Hölder's inequality

$$\begin{aligned}
\int_0^1 B_{i,k}(x)\,N_{i,k}(x)\,dx &\leqslant \int_{x_i}^{x_{i+k+1}} N_{i,k}(x) \left(\int_{x_i}^{x_{i+k+1}} M_{i,k}(t) \right. \\
&\qquad \times \left. \left(\int_{x_i}^{x_{i+k+1}} |f'(u)|^p\,du \right) dt \right) dx \\
&= \int_{x_i}^{x_{i+k+1}} N_{i,k}(x) \left(\int_{x_i}^{x_{i+k+1}} |f'(u)|^p\,du \right) dx \\
&\leqslant |\Delta_n| \int_{x_i}^{x_{i+k+1}} |f'(u)^p|\,du,
\end{aligned} \tag{7}$$

since

$$\int_{x_i}^{x_{i+k+1}} N_{i,k}(x)\,dx = \frac{x_{i+k+1}-x_i}{k+1} \leqslant |\Delta_n|.$$

Now, from (5), (6), and (7) we deduce

$$\begin{aligned}\|U_{n,k}f-f\|_p &\\ &\leqslant (k+1)^{1/q}\,|\Delta_n|^{1/q}|\Delta_n|^{1/p}\left(\sum_i \int_{x_i}^{x_{i+k+1}} |f'(u)|^p\,du\right)^{1/p}\\ &\leqslant (k+1)|\Delta_n|\left(\int_0^1 |f'(u)|^p\,du\right)^{1/p}.\end{aligned}$$

Q.E.D.

For $f \in L^p(I)$ and $0 \leqslant t \leqslant 1$, the K-functional of Peetre [20] is defined by

$$K_p(t, f) = \inf\{\|f-g\|_p + t\|g'\|_p,\ g \in L^{p,1}(I)\} \tag{8}$$

and the integral modulus of continuity by

$$\omega_{1,p}(f, h) = \sup_{0<t\leqslant h} \|f(\cdot+t)-f(\cdot)\|_p\,(I_t),$$

where $\|\cdot\|\ (I_t)$ means that the norm is to be taken over the interval $I_t = [0, 1-t]$. Johnen [13] proved that there exist constants $C_1 > 0$ and $C_2 > 0$, independent of f and p, such that:

$$C_1\omega_{1,p}(f, t) \leqslant K_p(t, f) \leqslant C_2\omega_{1,p}(f, t) \qquad (0 \leqslant t \leqslant 1) \tag{9}$$

THEOREM 2. *There exists a constant* $M_k > 0$ (*independent of* f *and* $1 \leqslant p \leqslant +\infty$) *such that, for all* $f \in L^p(I)$,

$$\|U_{n,k}f-f\|_p \leqslant M_k\omega_{1,p}(f, |\Delta_n|).$$

Proof. From Theorem 1(i), we have $\|U_{n,k}f-f\|_p \leqslant 2\|f\|_p$ for all $f \in L^p(I)$ and from Lemma 2, we have

$$\|U_{n,k}\,g-g\|_p \leqslant (k+1)\|g'\|_p\,|\Delta_n| \qquad \text{for all} \quad g \in L^{p,1}(I),$$

thus

$$\begin{aligned}\|U_{n,k}f-f\|_p &\leqslant \|U_{n,k}(f-g)-(f-g)\|_p + \|U_{n,k}\,g-g\|_p\\ &\leqslant 2\|f-g\|_p + (k+1)\,\|g'\|_p|\Delta_n|\\ &\leqslant 2(\|f-g\|_p + k|\Delta_n|\,\|g'\|_p).\end{aligned}$$

Taking the infimum over all $g \in L^{p,1}(I)$ and using (8) and (9) with $t = k|\Delta_n| \leqslant 1$, we get $M_k = 2kC_2$,

$$\|U_{n,k}f - f\|_p \leqslant (K_p(k|\Delta_n|, f) \leqslant 2C_2\omega_{1,p}(f, k|\Delta_n|) \leqslant 2kC_2\omega_{1,p}(f, |\Delta_n|).$$

III. Spectral Properties in $L^2(I)$

THEOREM 3. (i) *$U_{n,k}$ is a self-adjoint operator in $L^2(I)$ having $n+k$ real positive simple eigenvalues λ_j $(0 \leqslant j \leqslant n+k-1)$,*

$$0 < \lambda_{n+k-1} < \cdots < \lambda_j < \cdots < \lambda_1 < \lambda_0 = 1.$$

(ii) *The associated eigenfunctions $V_j(x)$ are defined by $V_j(x) = \sum_{i=-k}^{n-1} \omega_{ij} N_{i,k}(x)$ where $\tilde{V}_j = (\omega_{ij}, -k \leqslant i \leqslant n-1)$ is the* jth *eigenvector of the oscillatory matrix $A_{n,k} = (\langle M_{i,k}, N_{j,k}\rangle, -k \leqslant i, j \leqslant n-1)$. Moreover, $V_0(x) = 1$ and $S^-(V_j) \leqslant j$ for $1 \leqslant j \leqslant n+k-1$ (number of changes of sign on I).*

(iii) *The best least square approximation S of $f \in L^2(I)$ in $S_p(k, \Delta_n)$ is*

$$S(x) = \sum_{j=0}^{n+k-1} \gamma_j \langle f, V_j\rangle V_j(x)$$

where

$$\gamma_j^{-1} = \lambda_j \sum_{i=-k}^{n-1} \left(\frac{x_{i+k+1} - x_i}{k+1}\right) \omega_{ij}^2$$

and

$$\langle f, V_j\rangle = \sum_{i=-k}^{n-1} \omega_{ij} \mu_{i,k}(f),$$

where

$$\mu_{i,k}(f) = \langle f, N_{i,k}\rangle = \int_0^1 N_{i,k}(t) f(t)\, dt.$$

Proof. $U_{n,k}$ is an operator of finite rank $n+k$ and its restriction to $Sp(k, \Delta_n)$ has a $(n+k)\times(n+k)$ matrix in the B-spline basis:

$$A_{n,k} = (\langle M_{i,k}, N_{j,k}\rangle, -k \leqslant i, j \leqslant n-1).$$

It is *stochastic*, for $\sum_j \langle M_{i,k}, N_{j,k}\rangle = \langle M_{i,k}, e_0\rangle = 1$, hence $\lambda_0 = 1$ and $\tilde{V}_0 = (1, ..., 1) \in \mathbb{R}^{n+k}$ are associated eigenvalue and eigenvector, moreover

$|\lambda_j| \leqslant 1$ for $j \geqslant 1$. But $A_{n,k}$ is also the matrix $G = G_\infty$, given by de Boor in [5] and [6], which is *totally positive* (all subdeterminants are non negative). Since $\langle M_{i,k}, N_{i-1,k} \rangle$ and $\langle M_{i,k}, N_{i+1,k} \rangle$ are strictly positive for $k \geqslant 1$, Theorem 2 [12, p. 454] of Gantmakher and Krein implies that $A_{n,k}$ is an *oscillatory matrix* (i.e., some power of $A_{n,k}$ is strictly totally positive: all subdeterminants are positive). Therefore, its eigenvalues are *real*, *positive*, and *simple* and the jth eigenvector $\tilde{V}_j = (\omega_{ij}, -k \leqslant i \leqslant n-1)$ has exactly j *strict changes of sign*,

$$S^-(V_j) = S^+(V_j) = j \qquad (1 \leqslant j \leqslant n+k-1).$$

The jth eigenfunction of $U_{n,k}$ is of course $V_j(x) = \sum_{i=-k}^{n-1} \omega_{ij} N_{i,k}(x)$ (in particular $V_0(x) = \sum_i N_{i,k}(x) = 1$) and, in view of the variation-diminishing property of B-splines, we have $S^-(V_j) \leqslant j$ for $j \geqslant 1$.

The operator $U_{n,k}$ being self-adjoint in $L^2(I)$, its eigenfunctions V_j form an orthogonal basis in $Sp(k, \Delta_n)$. The orthogonal projection of $f \in L^2(I)$ on this space is

$$S(x) = \sum_{j=0}^{n+k-1} \gamma_j \langle f, V_j \rangle V_j(x),$$

where

$$\begin{aligned}
\gamma_j^{-1} &= \langle V_j, V_j \rangle = \sum_{l,m} \omega_{lj} \omega_{mj} \langle N_{l,k}, N_{m,k} \rangle \\
&= \sum_l \omega_{lj} \left(\frac{x_{l+k+1} - x_l}{k+1} \right) \left(\sum_m \langle M_{l,k}, N_{m,k} \rangle \omega_{mj} \right) \\
&= \sum_l \omega_{lj} \left(\frac{x_{l+k+1} - x_l}{k+1} \right) (A_{nk} \tilde{V}_j)_l = \lambda_j \sum_l \left(\frac{x_{l+k+1} - x_l}{k+1} \right) \omega_{lj}^2.
\end{aligned}$$

Let $\mu_{i,k}(f) = \langle f, N_{i,k} \rangle = \int_0^1 N_{i,k}(t) f(t)\, dt \quad (-k \leqslant i \leqslant n-1)$ be the "moments" of f w.r.t. the B-spline basis, then we have

$$\langle f, V_j \rangle = \sum_{i=-k}^{n-1} \omega_{ij} \langle f, N_{i,k} \rangle = \sum_{i=-k}^{n-1} \omega_{ij} \mu_{i,k}(f). \qquad \text{Q.E.D.}$$

Remark. From a practical point of view, once the numbers ω_{ij} (i.e., the components of the eigenvectors $\tilde{V}_j$ of $A_{n,k}$) and the γ_j have been computed, the only work to do is to compute the moments $\mu_{i,k}(f)$ and the linear combinations $\langle f, V_j \rangle$ to get the projection S of f.

(a) The approximate computation of $\mu_{i,k}(f)$ can be made using special Gaussian quadrature rules for the weight functions $N_{i,k}(x) \geqslant 0$ (see, e.g., [21]).

(b) The evaluation of $S(x)$ can be made in the B-spline basis

$$S=\sum_j \gamma_j\langle f, V_j\rangle V_j=\sum_j\left(\sum_j \omega_{ij}\gamma_j\langle f, V_j\rangle\right) N_i$$

using the classical De Boor-Cox algorithm.

IV. Piecewise Linear and Quadratic Orthogonal Splines

4.1. *Piecewise Linear Splines*

Let $U_{n,k}=U_n$ and $N_{i,k}=N_i$ for $k=1$. (hat-functions).

THEOREM 4. (i) *The* $(n+1)$ *eigenvalues* $\lambda_j^{(n)}$ *of* U_n *verify*

$$\lambda_n^{(n)}=1/3<\lambda_j^{(n)}<\lambda_0^{(n)}=1 \qquad (1\leqslant j\leqslant n-1).$$

(ii) *The* $(n+1)$ *orthogonal eigenfunctions* $V_j^{(n)}(x)$ *of* U_n *have exactly* j *real simple roots in* $(0, 1)$, *moreover,*

$$V_0^{(n)}(x)=\sum_{i=0}^{n} N_i(x)=1,$$

$$V_n^{(n)}(x)=\sum_{i=0}^{n} (-1)^i N_i(x).$$

(iii) *For the uniform partition* $\Delta_n=\{i/n, 0\leqslant i\leqslant n\}$, *we have*

$$\lambda_j^{(n)}=(2+\cos(j\pi/n))/3,$$

$$V_j^{(n)}=\sum_{i=0}^{n}\cos(ij\pi/n)\,N_i(x), \qquad 0\leqslant j\leqslant n$$

when $n\to+\infty$, $V_j^{(n)}$ *converges uniformly to* $\cos(j\pi x)$, *for every* $j\geqslant 0$ *fixed.*

Proof. The matrix A_n of U_n is

$$A_n=\frac{1}{3}\begin{bmatrix} 2 & 1 & 0\ldots\ldots\ldots\ldots & & \\ a_1 & 2 & b_1\ldots\ldots\ldots\ldots & & \\ 0 & a_2 & 2 \quad b_2\ldots\ldots\ldots & & \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & & \\ \ldots\ldots\ldots\ldots & a_{n-1} & 2 & b_{n-1} & \\ \ldots\ldots\ldots & 0 & 1 & 2 & \end{bmatrix}$$

where $a_i = h_i/(h_{i+1} + h_i)$, $b_i = 1 - a_i$, $h_i = x_i - x_{i-1}$. One verifies (i) by Gershgorin's theorem on eigenvalues and direct computation and (ii) by the fact that A_n is an oscillatory stochastic matrix (the principal minors are positive). Since $\tilde{V}_j^{(n)} = (\omega_{ij}, 0 \leqslant i \leqslant n)$ has exactly j changes of sign (Theorem 3), and since $V_j^{(n)}$ is piecewise linear, it has also exatly j simple roots in $(0, 1)$.

When Δ_n is uniform, $a_i = b_i = \frac{1}{2}$ and the eigenvalues and eigenvectors of A_n are known explicitly, which gives (iii). The uniform convergence of $V_j^{(n)}(x)$ to $V_j(x) = \cos(j\pi x)$ follows from the fact that $V_j^{(n)}$ is the piecewise linear interpolant of V_j at the points of Δ_n. ∎

EXAMPLE. For $n = 3$ and $\Delta_n = \{i/n, 0 \leqslant i \leqslant 3\}$,

$$\begin{aligned} \lambda_0 &= 1, & \tilde{V}_0^T &= (1, 1, 1, 1), \\ \lambda_1 &= \tfrac{5}{6}, & \tilde{V}_1^T &= (1, \tfrac{1}{2}, -\tfrac{1}{2}, -1), \\ \lambda_2 &= \tfrac{1}{2}, & \tilde{V}_2^T &= (1, -\tfrac{1}{2}, -\tfrac{1}{2}, 1), \\ \lambda_3 &= \tfrac{1}{3}, & \tilde{V}_3^T &= (1, -1, 1, -1). \end{aligned}$$

Let us compute successively (Theorem 3(iii)):

$$\gamma_0^{-1} = \langle V_0, V_0 \rangle = 1, \qquad \gamma_1^{-1} = \langle V_1, V_1 \rangle = \tfrac{5}{12},$$
$$\gamma_2^{-1} = \langle V_2, V_2 \rangle = \tfrac{1}{4}, \qquad \gamma_3^{-1} = \langle V_3, V_3 \rangle = \tfrac{1}{3}$$

and the matrix $C = (C_{ij})$, where

$$\begin{aligned} C_{ij} = \gamma_j \sum_k \omega_{ki} \omega_{kj} &= 1 + \tfrac{12}{5} \cos(i\pi/3) \cos(j\pi/3) \\ &\quad + 4 \cos(2i\pi/3) \cos(2j\pi/3) + 3 \cos(i\pi) \cos(j\pi), \end{aligned}$$

we get

$$C = \frac{2}{5} \begin{bmatrix} 26 & -7 & 2 & -1 \\ -7 & 14 & -4 & 2 \\ 2 & -4 & 14 & -7 \\ -1 & 2 & -7 & 26 \end{bmatrix}.$$

Let $\mu_i(f) = \int_0^1 N_i(t) f(t)\, dt$ and

$$\alpha_0 = \tfrac{2}{5}(26\mu_0 - 7\mu_1 + 2\mu_2 - \mu_3), \qquad \alpha_1 = \tfrac{2}{5}(-7\mu_0 + 14\mu_1 - 4\mu_2 + 2\mu_3),$$
$$\alpha_2 = \tfrac{2}{5}(2\mu_0 - 4\mu_1 + 14\mu_2 - 7\mu_3), \qquad \alpha_3 = \tfrac{2}{5}(-\mu_0 + 2\mu_1 - 7\mu_2 + 26\mu_3).$$

Then, the orthogonal projection of f on $Sp(1, \Delta_3)$ is

$$S(x) = \sum_{i=0}^{3} \alpha_i N_i(x),$$

and it is easily verified that $\|S\|_\infty \leqslant 3 \|f\|_\infty$ (which is also true for every partition, see De Boor [7], Ciesielski [3]).

Remark. The above result (iii) can be extended to more general partitions. For $\alpha > -1$ and $1 \leqslant i \leqslant n-1$ define

$$a_i = \frac{2\alpha + i - 1}{2(\alpha + i)} \qquad \text{and} \qquad b_i = 1 - a_i = \frac{i+1}{2(\alpha+i)}.$$

In order that $a_i = h_i/(h_i + h_{i+1})$, we must take the partition Δ_n of I defined by

$$h_1^{-1} = 1 + \frac{b_1}{a_1} + \frac{b_1 b_2}{a_1 a_2} + \cdots + \frac{b_1 \cdots b_{n-1}}{a_1 \cdots a_{n-1}},$$

$$h_i = h_1 \cdot \frac{b_1 \cdots b_{i-1}}{a_1 \cdots a_{i-1}} \qquad \text{for} \quad 2 \leqslant i \leqslant n.$$

Define the polynomials $\tilde{C}_n^{(\alpha)}(x)$ by the recurrence relation of ultraspherical polynomials

$$x\tilde{C}_n^{(\alpha)}(x) = a_i \tilde{C}_{n-1}^{(\alpha)}(x) + b_i \tilde{C}_{n+1}^{(\alpha)}(x), \tag{10}$$

but with different initial conditions

$$\tilde{C}_0^{(\alpha)}(x) = 1 \qquad \text{and} \qquad \tilde{C}_1^{(\alpha)}(x) = x.$$

From (10), we deduce the eigenvalues

$$\lambda_k^{(n)} = (2 + x_k^{(n)})/3,$$

and the eigenvectors

$$\tilde{v}_k^{(n)} = (\tilde{C}_0^{(\alpha)}(x_k^{(n)}), \tilde{C}_1^{(\alpha)}(x_k^{(n)}), ..., \tilde{C}_n^{(\alpha)}(x_k^{(n)})),$$

where $\{x_k^{(n)}, 0 \leqslant k \leqslant n\}$ are the $n+1$ roots in I of the equation

$$\tilde{C}_{n+1}^{(\alpha)}(x) = \tilde{C}_{n-1}^{(\alpha)}(x).$$

Numerical experiments suggest that, for all $k \geqslant 0$, the piecewise linear spline $v_k^{(n)}(x)$, whose value at $t_i = h_1 + \cdots + h_i$ is $\tilde{C}_i^{(\alpha)}(x_k^{(n)})$, converges uniformly, when $n \to +\infty$, to some special function $v_k(x)$. Magnus [15] proved that, at least near $x=1$, $v_k(x)$ has the behavior of $F(j'_k x^{1/(3-2\alpha)})$ $(\alpha < \frac{3}{2})$, where $\{j'_k, k \geqslant 0\}$ are the abscissae of the extrema of $F(x) = \Gamma(3/2-\alpha)(2/x)^{1/2-\alpha} J_{1/2-\alpha}(x)$. For $\alpha = 1$, $J_{-1/2}(x) = \sqrt{2/\pi}(\cos x/\sqrt{x})$, $j'_k = k\pi$, and $v_k(x) = \cos(k\pi x)$. For $\alpha = \frac{1}{2}$, j'_k is a zero of J_1 and $v_k(x) = J_0(j'_k \sqrt{x})$ (orthogonal system of Fourier–Bessel). I conjecture that $v_k^{(n)}(x)$ converges uniformly to $v_k(x)$ in the whole interval I.

4.2. *Quadratic Splines on an Uniform Partition*

The matrix A_n has rank $n+2$ and is given explicitly in [5]. For example,

$$A_5 = \frac{1}{120} \begin{bmatrix} 72 & 42 & 6 & 0 & 0 & 0 & 0 \\ 21 & 60 & 37.5 & 1.5 & 0 & 0 & 0 \\ 2 & 25 & 66 & 26 & 1 & 0 & 0 \\ 0 & 1 & 26 & 66 & 26 & 1 & 0 \\ 0 & 0 & 1 & 26 & 66 & 26 & 1 \\ 0 & 0 & 0 & 1.5 & 37.5 & 60 & 21 \\ 0 & 0 & 0 & 0 & 6 & 42 & 72 \end{bmatrix}.$$

This matrix is a perturbation of

$$A_5^* = \frac{1}{120} \begin{bmatrix} 66 & 52 & 2 & 0 & 0 & 0 & 0 \\ 26 & 67 & 26 & 1 & 0 & 0 & 0 \\ 1 & 26 & 66 & 26 & 1 & 0 & 0 \\ 0 & 1 & 26 & 66 & 26 & 1 & 0. \\ 0 & 0 & 1 & 26 & 66 & 26 & 1 \\ 0 & 0 & 0 & 1 & 26 & 67 & 26 \\ 0 & 0 & 0 & 0 & 2 & 52 & 66 \end{bmatrix}$$

For $0 \leqslant k \leqslant n+1$, the eigenvalues of A_n^* are

$$\mu_k^{(n)} = ((x_k^{(n)})^2 + 13x_k^{(n)} + 16)/30, \tag{11}$$

where $x_k^{(n)} = \cos(k\pi/(n+1))$, and the associated eigenvectors are

$$\tilde{\mu}_k^{(n)} = (1, x_k^{(n)}, T_2(x_k^{(n)}), ..., T_{n+1}(x_k^{(n)})), \tag{12}$$

where $T_n(x)$ is the Chebyshev polynomial of degree n. (This is a straightforward consequence of the recurrence relation $2xT_n(x) = T_{n-1}(x) + T_{n+1}(x)$). We can use (11) and (12) as starting eigenvalues and eigenvectors for the inverse power method applied to A_n. Numerical experiments show that the convergence is rather fast (about 15 iterations for eight digits). Moreover, we have, when $n \to +\infty$,

$$\lambda_1^{(n)} < \mu_1^{(n)} \sim 1 - \frac{\pi^2}{4(n+1)^2}$$

($n = 5$: $\lambda_1 = 0.909679116 < \mu_1 = 0.933611008$).

The least eigenvalue $\lambda_{n+1}^{(n)}$ is very near $\mu_{n+1}^{(n)} = \frac{2}{15}$ (e.g., $n = 5$, $\lambda_6 = 0.123328549 < \mu_6 = 0.133\,333\,333$, $n = 8$, $\lambda_9 = 0.126779075 < \mu_9 = 0.133\,333\,333$). Moreover the drawing of graphs suggest, as in the piecewise linear case, the uniform convergence of $v_k^{(n)}(x)$ to $v_k(x) = \cos(k\pi x)$ when $n \to +\infty$. Of course these conjecture have to be proved. Similar results are observable for cubic orthogonal splines.

V. Some Extensions of the Results

5.1. It has been proved [23, Chap. 4] and [24]) that not only $S^-(V_j) \leqslant j$ (see Theorem 3(ii)) but that, even for $k \geqslant 2$, V_j has exactly j simple real roots in (0, 1) and that the roots of V_{j+1} lie between those of V_j (like the roots of orthogonal polynomials).

5.2. When $\langle f, g \rangle = \int_0^1 \omega(t) f(t) g(t)\, dt$ with a positive weight function (essentially $\omega(t) = t^\alpha (1-t)^\beta$; $\alpha, \beta > -1$) the results of Theorems 1 and 2 are valid with minor modifications. The results of Theorem 3 would be valid if one should be able to prove that the corresponding matrix $A_{n,k}$ is yet an oscillatory matrix. The case $\omega(t) = t^\alpha e^{-t}$ on $\mathbb{R}^+$ is also interesting. These scalar products give rise to Jacobi and Laguerre splines.

5.3. In a similar way, Chebyshev B-splines [27] could be used to define orthogonal generalized splines, but the main problem also concerns the matrix $A_{n,k}$.

5.4. The extension to tensor-product orthogonal splines on $Q = I \times I$ is straightforward. More generally, if Ω is a triangulated domain in $\mathbb{R}^2$ and if there exists a basis of positive B-splines $\{N_j\}$ forming a partition of unity on Ω, then it is possible to define the operator:

$$Uf(x) = \sum_j \left(\int_\Omega M_j(t) f(t)\, dt \right) N_j(x)$$

and the corresponding orthogonal splines as eigenfunctions of this operator if its matrix w.r.t. the basis $\{N_j\}$ has good properties.

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